

NON-SEMIGROUP GRADINGS OF ASSOCIATIVE ALGEBRAS

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ABSTRACT. It is known that there are Lie algebras with non-semigroup gradings, i.e. such that the binary operation on the grading set is not associative. We provide a similar example in the class of associative algebras.

INTRODUCTION

Let A be a (generally, not necessary associative, or Lie, or satisfying any other distinguished identity) algebra, and $A = \bigoplus_{g \in \Gamma} A_g$ its grading over a grading set $(\Gamma, *)$, i.e. $*$: $\Gamma \times \Gamma \rightarrow \Gamma$ is a partial binary operation defined for each pair (g, h) such that $A_g A_h \neq 0$, in which case $A_g A_h \subseteq A_{g*h}$. How the identities satisfied by the algebra A are related to identities satisfied by the grading set Γ ? Since the operation $*$ on Γ is partial, in the latter case it makes sense to speak about an (im)possibility to complete $*$ in such a way that it will satisfy that or another identity, or, in more strict terms, about an (im)possibility of embedding of $(\Gamma, *)$ into an appropriate magma[†] (G, \cdot) such that $g * h = g \cdot h$ whenever $A_g A_h \neq 0$.

It is immediate that commutativity or anticommutativity of A implies that Γ can be embedded into a commutative magma. Elementary manipulations involving homogeneous components A_g 's of graded Lie and associative algebras may suggest that both Jacobi identity and associativity of the algebra A are strongly connected with the associativity of the grading set Γ . In the Lie case, it was believed for a while (and even claimed in an influential paper [PZ] as Theorem 1(a)) that each grading of a Lie algebra is a *semigroup grading*, i.e. the grading set $(\Gamma, *)$ can be embedded into a semigroup. This is indeed so for all gradings of Lie and associative algebras appearing naturally (root space decompositions with respect to Cartan subalgebra, gradings arising from various group or Hopf algebra actions on the algebra, \mathbb{Z} -gradings providing connection between Lie and Jordan algebras, semigroup algebras and their twisted variants, grading by Pauli matrices motivated by physics, etc.). However, in [E1] and [E2] examples of non-semigroup gradings of Lie algebras were given. The aim of this note is to provide example of a non-semigroup grading of an associative algebra. This is done in §1 following approach of [Z, §3], where it was shown how non-semigroup gradings of Lie algebras can be constructed using δ -derivations. §2 contains some further questions.

1. AN EXAMPLE OF NON-SEMIGROUP GRADING

In the associative case, instead of δ -derivations we may consider a slightly more general notion of (δ, γ) -derivations, i.e. linear maps $D : A \rightarrow A$ on an algebra A such that

$$D(xy) = \delta D(x)y + \gamma x D(y)$$

for any $x, y \in A$, and some fixed elements of the ground field δ, γ . In the Lie case, due to anticommutativity, any such condition implies that either $\delta = \gamma$, i.e. D is a δ -derivation, or that D is “generalized centroid”, i.e.

$$D(xy) = (\delta + \gamma)D(x)y = (\delta + \gamma)x D(y)$$

for any $x, y \in A$, the latter condition being too restrictive to be interesting. (The same dichotomy holds for commutative algebras).

Date: September 13, 2016.

[†] “Magma” means a set with an (everywhere defined) binary operation on it, without any additional conditions. In the older literature, the term “groupoid” was used instead, but since then the latter term was taken by category theorists.

Lemma 1. *Let A be a finite-dimensional algebra over an algebraically closed field K , and $A = \bigoplus_{\lambda \in K} A_\lambda$ is the root space decomposition with respect to an (δ, γ) -derivation of A . Then $A_\lambda A_\mu \subseteq A_{\delta\lambda + \gamma\mu}$ for any $\lambda, \mu \in K$.*

(Note that the algebra A here is not necessary associative).

Proof. It is trivial to check that if x and y are eigenvectors of an (δ, γ) -derivation of A , corresponding to eigenvalues λ and μ respectively, then the product xy is an eigenvector corresponding to $\delta\lambda + \gamma\mu$ (or zero, if $\delta\lambda + \gamma\mu$ is not an eigenvalue). Then proceed by induction on the sum of multiplicities of the respective eigenvalues, exactly the same way as in, for example, [J, Chapter III, §2]. \square

The following is a slightly modified “nonassociative” analogue of the Lie-algebraic statement [Z, Proposition 3.1].

Proposition. *Let A be a finite-dimensional algebra over an algebraically closed field, and D an (δ, γ) -derivation of A . Suppose that there are roots $\lambda, \mu, \eta, \theta, \xi$ (not necessarily distinct) in the root space decomposition of A with respect to D such that*

- (1) $0 \neq A_\lambda A_\eta \subseteq A_\theta, \quad A_\theta A_\mu \neq 0,$
- (2) $0 \neq A_\eta A_\mu \subseteq A_\xi, \quad A_\lambda A_\xi \neq 0,$

and $(\delta^2 - \delta)\lambda \neq (\gamma^2 - \gamma)\mu$. Then the said root space decomposition is a non-semigroup grading of A .

(Again, the algebra A here is not assumed to be associative, or Lie, or to satisfy any other distinguished identity. Note also that the conditions (1) and (2) are somewhat weaker than $(A_\lambda A_\eta)A_\mu \neq 0$ and $A_\lambda(A_\eta A_\mu) \neq 0$, respectively).

Proof. The conditions (1) and (2) ensure that both expressions $(\lambda * \eta) * \mu$ and $\lambda * (\eta * \mu)$ are defined. If the root space decomposition of A with respect to D is a semigroup grading, then these two expressions are equal: $(\lambda * \eta) * \mu = \lambda * (\eta * \mu)$. By Lemma 1, this equality is equivalent to $(\delta^2 - \delta)\lambda = (\gamma^2 - \gamma)\mu$, a contradiction. \square

Corollary. *The conclusion of Proposition holds in each of the following cases:*

- (i) $\delta = \gamma \neq 0, 1$, and $\lambda \neq \mu$;
- (ii) $\delta \neq \gamma$, $\delta + \gamma \neq 1$, and $\lambda = \mu \neq 0$.

Proof. Obvious. \square

Now we will provide an example of a family of associative algebras having δ -derivations as in heading (i) of the Corollary, and hence admitting a non-semigroup grading. Let V be a vector space over a field K , and $f_L, f_R, g_L, g_R : V \rightarrow V$ be four linear maps. Consider the vector space direct sum

$$Ka \oplus Ke \oplus V \oplus V',$$

where Ka and Ke are one-dimensional vector spaces spanned by elements a and e respectively, and V' is a second copy of V , identified with V via a nondegenerate linear map $v \mapsto v'$. Define the multiplication on this direct sum as follows:

$$a^2 = 0, \quad e^2 = e, \quad av = f_L(v)', \quad va = f_R(v)', \quad av' = g_L(v), \quad v'a = g_R(v),$$

where $v \in V$, and the rest of the products between basic elements is zero. The associativity of the so defined algebra, let us denote it as $A(f_L, f_R, g_L, g_R)$, is equivalent to the following conditions:

$$\begin{aligned} f_L \circ g_L &= g_L \circ f_L = 0 \\ f_R \circ g_R &= g_R \circ f_R = 0 \\ g_R \circ f_L &= g_L \circ f_R \\ f_R \circ g_L &= f_L \circ g_R. \end{aligned}$$

Lemma 2. Suppose that each of the maps f_L, f_R, g_L, g_R is nonzero, and $(\delta, \gamma) \neq (0, 0)$. Then each (δ, γ) -derivation D of the algebra $A(f_L, f_R, g_L, g_R)$ is of the following form:

$$\begin{aligned} D(a) &= \alpha a + v_a + w'_a \\ D(e) &= \begin{cases} 0 & \text{if } \delta + \gamma \neq 1 \\ \beta e & \text{if } \delta + \gamma = 1 \end{cases} \\ D(v) &= \varphi(v) + \psi(v)', \quad v \in V \\ D(v') &= \tilde{\varphi}(v) + \tilde{\psi}(v)' \end{aligned}$$

where $\alpha, \beta \in K$, $v_a, w_a \in V$, $\varphi, \tilde{\varphi}, \psi, \tilde{\psi} : V \rightarrow V$ are linear maps, and the following conditions are satisfied:

$$\begin{aligned} (\delta f_R + \gamma f_L)(v_a) &= 0 \\ (\delta g_R + \gamma g_L)(w_a) &= 0 \end{aligned}$$

and

$$\begin{aligned} \tilde{\varphi} \circ f_L &= \gamma g_L \circ \psi \\ \tilde{\psi} \circ f_L &= \delta \alpha f_L + \gamma f_L \circ \varphi \\ \tilde{\varphi} \circ f_R &= \delta g_R \circ \psi \\ \tilde{\psi} \circ f_R &= \gamma \alpha f_R + \delta f_R \circ \varphi \\ \varphi \circ g_L &= \delta \alpha g_L + \gamma g_L \circ \tilde{\psi} \\ \psi \circ g_L &= \gamma f_L \circ \tilde{\varphi} \\ \varphi \circ g_R &= \gamma \alpha g_R + \delta g_R \circ \tilde{\psi} \\ \psi \circ g_R &= \delta f_R \circ \tilde{\varphi}. \end{aligned}$$

Proof. Direct calculations. □

The non-vanishing conditions of Lemma 2 are merely technical ones, to avoid consideration of numerous degenerate tedious cases.

We may specialize this setup in many different ways to get an example of an algebra having a (δ, γ) -derivation satisfying the condition of Proposition or its Corollary, and hence admitting a non-semigroup grading. One of the easiest ways is to set $f_L = f_R = g_L = g_R = f$, where $f \circ f = 0$ (say, V is 2-dimensional, and f has the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the canonical basis), $\delta = \gamma = -1$, $\alpha = \beta = 0$, $v_a = w_a = 0$, and $\psi = \tilde{\varphi} = 0$, $\varphi = \text{id}_V$, $\tilde{\psi} = -\text{id}_V$. Then D from Lemma 2 is a (-1) -derivation (or, *antiderivation*) of the algebra $A(f, f, f, f)$. The eigenvalues of D are $0, 1, -1$, with eigenspaces $A_0 = Ke \oplus Ka$, $A_1 = V$, and $A_{-1} = V'$. Then by heading (i) of Corollary, the root space decomposition $A(f, f, f, f) = A_0 \oplus A_1 \oplus A_{-1}$ is a non-semigroup grading. This fact can be also verified directly: as $A_0^2 = Ke \subset A_{-1}$, $A_0 A_1 = A_1 A_0 = (\text{Im } f)' \subset A_{-1}$, and $A_0 A_{-1} = A_{-1} A_0 = \text{Im } f \subset A_1$, we have the following (partial) operation on the grading set:

$$0 * 0 = 0, \quad 0 * 1 = 1 * 0 = -1, \quad 0 * (-1) = (-1) * 0 = 1,$$

what contradicts associativity:

$$1 = 0 * (-1) = (0 * 0) * (-1) \neq 0 * (0 * (-1)) = 0 * 1 = -1.$$

(Note that this is the same non-associative grading set as in the Lie-algebraic example in [E2]).

The algebra $A(f, f, f, f)$ is, obviously, commutative, with a commutative grading. By modifying this example to make the maps f_L, f_R, g_L, g_R different, it is possible to get various examples of associative non-commutative algebras with a non-semigroup grading, commutative or not. The relevant calculations are trivial, but somewhat cumbersome, and left to the interested reader.

2. FURTHER QUESTIONS

If $L = \bigoplus_{g \in \Gamma} L_g$ is a Lie algebra graded by an *abelian group* Γ , then its universal enveloping algebra $U(L)$ is a Γ -graded associative algebra, with the graded components $U(L)_g$ linearly spanned by monomials of the form $x_1 \dots x_k$, where $x_i \in L_{g_i}$ and $\sum_{i=1}^k g_i = g$ (see, e.g., [SF, Theorem 4.3]).

The algebra $U(L)$ is infinite-dimensional, what, perhaps, is not that interesting in our context. In the positive characteristic it is possible, however, to define the same grading on the finite-dimensional *restricted* universal enveloping algebra of a graded restricted Lie algebra. However, the facts that multiplication in Γ is defined everywhere, and is associative, are crucial in this construction, and it is unclear how to extend or modify it to grading by an arbitrary set Γ .

Question 1. *Is it possible to construct a grading of the (restricted) universal enveloping algebra, given (arbitrary, not necessary semigroup) grading of the underlying Lie algebra?*

A positive answer to this question will produce a plethora of non-semigroup gradings of finite-dimensional associative algebras in positive characteristic, different from those exhibited in §1: take any of the examples from [E1] or [E2] over a field of characteristic $p > 0$, pass, if necessary, to the p -envelope, and consider the restricted universal enveloping algebra.

Question 2. *What is the minimal dimension of an associative algebra admitting a non-semigroup grading?*

It is, probably, possible to prove, following the approach of [E2, Theorem in §1], and classification of low-dimensional associative algebras, that any grading of an associative algebra of dimension ≤ 3 is a semigroup grading. Since the underlying algebra is not necessary commutative, there are apriori much more possibilities for a noncommutative partial operation on a 2- and 3-element grading set. The relevant calculations are straightforward, but definitely cumbersome.

We also failed to find examples of non-semigroup gradings of associative algebras of dimension 4 and 5. The minimal dimension of an algebra with non-semigroup grading following the scheme of §1 is 6.

By analogy with question about gradings of simple Lie algebras from [E1], one may ask

Question 3. *Is it true that any grading of a full matrix algebra is a semigroup grading?*

Note that this question cannot be approached by constructing an appropriate (δ, γ) -derivation as in §1: it is easy to see that any (δ, γ) -derivation of a full matrix algebra is either an (inner) derivation, or a scalar multiple of the identity map (see, e.g., [S, Theorem 1] for a slightly more general statement).

Finally, note that, in principle, the same approach as in §1 may be used to construct examples of non-semigroup gradings in varieties of algebras satisfying other identities of degree 3 (like Leibniz, Zinbiel, left-symmetric, Lie-admissible, Alia algebras, etc.). Another interesting topic would be to explore the question from the point of view of operadic Koszul duality: for example, does the presence/absence of non-semigroup gradings of algebras over a binary quadratic operad \mathcal{P} entails the same for algebras over the operad Koszul dual to \mathcal{P} ?

ACKNOWLEDGEMENT

Thanks are due to Miroslav Korbelař for asking questions which prompted me to write this note.

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